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On a Graphical Method for determining the Orbit of a Binary Star. By Professor S. Glasenapp.

(*Communicated by Herbert Sadler.*)

The problem of determining the elements of a binary star consists of two separate parts: (1) to obtain the apparent orbit from a series of given observations, and (2) to compute the elements of the true orbit, the apparent orbit being given. Both parts can be solved by graphical and analytical methods.

Between the true and apparent orbit exists a geometrical relation which furnishes the means for obtaining one orbit when the other is given; this is a purely geometrical problem.

The apparent orbit is never known; we can only determine the probable apparent orbit of the component stars. Each computer tries to obtain the most probable orbit.

The general equation of an ellipse, being a curve of the second degree, is—

$$\alpha x + \beta y + \gamma x^2 + \delta xy + \epsilon y^2 + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x and y are the rectangular co-ordinates of the *comes* relative to the principal star which coincides with the origin of the co-ordinates. The plane of co-ordinates is perpendicular to the line of sight.

The coefficients of the equation (1) for an ellipse must satisfy the well-known conditions—

$$\begin{gathered}(\beta\delta - 2\alpha\epsilon)^2 - (\delta^2 - 4\epsilon\gamma)(\beta^2 - 4\epsilon) > 0, \\ \delta^2 - 4\epsilon\gamma < 0.\end{gathered} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

I should add to them the two following:—

$$\gamma < 0 \text{ and } \epsilon < 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which represents that the principal star lies in the inner part of the ellipse—i.e. that the ellipse obtained intersects the two co-ordinate axes of x and y on opposite sides of the principal star. If the conditions (3) are not fulfilled all the parts of the ellipse lie outside the principal star. In this case, in determining the elements of the true orbit by the usual methods, we shall obtain imaginary elements.

Let us now consider a very simple method for determining the coefficients α, β, γ, δ, and ϵ of the general equation (1).

Let us assume that the apparent ellipse is drawn with such care as is generally employed in graphical methods.

If we put $y=0$, the equation (1) will be reduced to an equation of the second degree in x—

$$\alpha x + \gamma x^2 + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whose roots are the co-ordinates of the two points where the

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apparent ellipse intersects the axis of x. These roots being x_1 and x_2, we obtain

$$\alpha = -\frac{x_1 + x_2}{x_1 x_2};\ \gamma = \frac{1}{x_1 x_2} \quad \dots\dots\dots \quad (5)$$

If, further, we put $x=0$, we obtain an analogous equation in y—

$$\beta y + \epsilon y^2 + 1 = 0, \quad \dots\dots\dots \quad (6)$$

whose roots are the co-ordinates of the two points where the apparent ellipse intersects the axis of y. These roots being y_1 and y_2, we obtain

$$\beta = -\frac{y_1 + y_2}{y_1 y_2} \qquad \epsilon = \frac{1}{y_1 y_2}{}^{*} \quad \dots\dots\dots \quad (7)$$

If, therefore, we measure with compasses the co-ordinates x_1, x_2, y_1, and y_2 of the four intersecting points of the apparent ellipse with the co-ordinate axes, we obtain immediately, by means of the formulæ (5) and (7), the values of the four coefficients α, β, γ, and ϵ.

For the determination of δ we must take a fifth point. Let its co-ordinates be x_3 and y_3; then, from the general equation (1), we obtain

$$\delta = -\frac{1 + \alpha x_3 + \beta y_3 + \gamma x_3^2 + \epsilon y_3^2}{x_3 y_3} \quad \dots\dots\dots \quad (8)$$

From the form of this equation we can see that the most accurate determination of δ will be obtained in the case when the divisor $x_3 y_3$ has its maximum value. It is always possible to choose, without any calculation, such a point on the apparent ellipse whose co-ordinates very nearly satisfy this condition. As a check we can take a second point on the opposite side of the apparent ellipse, and obtain a second value of δ.

The mean of both values will be more accurate and more free from accidental errors which may have been made in drawing the apparent ellipse.

It is easy to see that the calculation of the coefficients α, β, γ, δ, and ϵ by means of the formulæ (5), (7), and (8) is very simple. The results are quite independent of the errors which may occur in drawing secondary ellipses, circles, and lines. This is an advantage which is worthy of attention. Further, all the calculation can be made in 10 to 15 minutes of time.

To illustrate this method, I will here give a numerical example. I take the excellent "Handbook of Double Stars" of Edw. Crossley, F.R.A.S., Joseph Gledhill, F.R.A.S., and James M. Wilson, M.A., F.R.A.S., and select the example of a graphical determination of the orbit of *Castor*.

On Plate II., p. 116, of this "Handbook," we find the apparent ellipse of *Castor*. From this drawing we take the following co-ordinates of the four points of intersection of the apparent

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* The co-ordinates y_1 and y_2, and also x_1 and x_2, must have opposite sings; therefore γ and ϵ must be negative.

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ellipse with the axes of x and y. The positive end of the axis of x is directed to the north ($\theta = 0°$), the positive end of the axis of y is directed to the point whose position is determined by $\theta = 90°$.

We have—

$$\text{for } y = 0 \qquad x_1 = +111^{\text{mm}}\cdot 8 \qquad x_2 = -233^{\text{mm}}\cdot 0,$$
$$\text{for } x = 0 \qquad y_1 = +170^{\text{mm}}\cdot 4 \qquad y_2 = -115^{\text{mm}}\cdot 6,$$

whence we obtain—

$$\log \alpha = \log \left[-\frac{x_1 + x_2}{x_1 x_2} \right] = 7\cdot 6676_n - 10, \quad \log \beta = \log \left[-\frac{y_1 + y_2}{y_1 y_2} \right] = 7\ 4444 - 10$$

$$\log \gamma = \log \left[\frac{1}{x_1 x_2} \right] = 5\cdot 5842_n - 10, \qquad \log \epsilon = \log \left[\frac{1}{y_1 y_2} \right] = 5\cdot 7056_n - 10.$$

To obtain δ we choose a point which is determined by the co-ordinates—

$$x_3 = -200^{\text{mm}} \qquad y_3 = -75^{\text{mm}},$$

which gives us—

$$\log \delta = \log \left[-\frac{1 + \alpha x_3 + \beta y_3 + \gamma x_3^2 + \epsilon y_3^2}{x_3 y_3} \right] = 4\cdot 8230 - 10.$$

To check this we take another point on the opposite side of the ellipse, whose co-ordinates are—

$$x_4 = +50^{\text{mm}} \qquad y_4 = +150^{\text{mm}},$$

and thus obtain a second value of $\log \delta = 4\cdot 8549$. The logarithm of the mean of these two values of δ is equal to—

$$\log \delta = 4\cdot 8392 - 10.$$

Thus we have obtained the values of the coefficients α, β, γ, δ, and ϵ of the equation (1).

This example is a good proof of the facility with which the coefficients α, β, γ, &c. can be obtained by the proposed method.

In investigating the true orbit I always use the very elegant formulæ of the late Russian Professor of the Imperial University at Kazan, M. Kowalsky,* which give the following relations between the elements of the true orbit and the coefficients α, β, γ, &c., of the general equation (1):—

$$\left.\begin{aligned}
\frac{\tan^2 i}{q^2} \,.\, \sin 2\Omega &= \delta - \tfrac{1}{2}\alpha\beta \\
\frac{\tan^2 i}{q^2} \,.\, \cos 2\Omega &= (\gamma - \epsilon) - \tfrac{1}{4}(\alpha^2 - \beta^2). \\
\frac{2}{q^2} + \frac{\tan^2 i}{q^2} &= -(\gamma + \epsilon) + \tfrac{1}{4}(\alpha^2 + \beta^2). \\
e \sin \lambda &= -\frac{q}{2}(\beta \cos \Omega - \alpha \sin \Omega) \cos i. \\
e \cos \lambda &= -\frac{q}{2}(\beta \sin \Omega + \alpha \cos \Omega). \\
a &= \frac{q}{1 - e^2}.
\end{aligned}\right\} \quad \ldots\ldots \quad (9)$$

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* These formulæ may be found in the *Proceedings of the Kazan Imp. University* for the year 1873.

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where—

i is the inclination of plane of the orbit to the line of sight (plane of xy).

Ω is the position-angle of the node.

e is the eccentricity.

λ is the angle between the node and the periastron $(\pi - \Omega)$.

a is the semi-axis major.

The first and second equations determine the node Ω and $\frac{\tan^2 i}{q^2}$; from the third we obtain $\frac{2}{q^2}$, and hence q and i; then the fourth and fifth equations determine e and λ, and from the last one we obtain the value of a.

The calculation of the elements by these formulæ, as may be seen, is very simple. The geometrical elements can be obtained in half an hour when α, β, γ, δ, and ϵ are given.

For example, we deduce—

$$\log \frac{\tan^2 i}{q^2} . \sin 2\Omega = 5{\cdot}1264$$

$$\log \frac{\tan^2 i}{q^2} . \cos 2\Omega = 4{\cdot}9464$$

$$\log \tan 2\Omega = 0{\cdot}1768$$

$$2\Omega = 56^\circ{\cdot}35 \qquad \Omega = 28^\circ{\cdot}18$$

$$\log \frac{\tan^2 i}{q^2} = 5{\cdot}2060$$

Further—

$$\log \left[\frac{2}{q^2} + \frac{\tan^2 i}{q^2} \right] = 5{\cdot}9846$$

and hence—

$$\log \frac{2}{q^2} = 5{\cdot}9055$$

$$\log q = 2{\cdot}1978$$

$$\log \tan^2 i = 9{\cdot}6015 \qquad i = 32^\circ{\cdot}30$$

$$\log e \sin \lambda = 9{\cdot}4911_n$$

$$\log e \cos \lambda = 9{\cdot}3420$$

$$\log \tan \lambda = 0{\cdot}1491_n \qquad \lambda = 305^\circ{\cdot}35$$

$$\log e = 9{\cdot}5901 \qquad \epsilon = 0{\cdot}380$$

and finally—

$$\log a = \log \frac{q}{1 - e^2} = 2{\cdot}2816 \qquad a = 184^{\text{mm}}{\cdot}3$$

To convert the value of a into seconds of arc, we take from the "Handbook," on p. 113, the distances on the millimetre scale. We find there that—

$$1238^{\text{mm}}{\cdot}2 = 45''{\cdot}07$$

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Therefore—

$$a = 184^{\mathrm{mm}}\cdot3 = 6''\cdot71.$$

Let us compare the geometrical elements of *Castor* thus obtained with those which are found by the authors of the "Handbook":—

Elements of the Handbook.	Glasenapp's Elements.
$a = 6''\cdot67$	$a = 6''\cdot71$
$e = 0\cdot38$	$e = 0\cdot38$
☊ $= 28^\circ\cdot25$	☊ $= 28^\circ\cdot18$
$i = 32\cdot25$	$i = 32\cdot30$
$\lambda = 305\cdot17$	$\lambda = 305\cdot35$

The elements T and n must be determined by one of the usual methods.

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On the Determination of Normal Places. By Lieut-General J. F. Tennant, R.E., F.R.S.

When it is proposed to correct an approximate orbit by comparing the places deduced from it with a large number of observations, and obtain corrections to the elements by the method of least squares, each observated element of the place would produce an equation of condition. It becomes then necessary to reduce the number of these equations of condition somehow, so as to limit the amount of computation in forming normal equations.

It seems to have been the practice to compute the equation of condition for each observation, and then to take the mean of the equations corresponding to a group of observations, and use it as one equation. Now the use of normal places formed by comparing the observations themselves with an ephemeris from approximate elements is practically universal. Of course the accuracy of these normals is vital. No care in computing can produce trustworthy results unless the normals are accurate, and to make them so a considerable number of observations have to be used, which probably extend over a considerable time. The determination of normal places has been treated of in the fourth volume of Oppolzer's *Lehrbuch*, but his remarks do not seem to me to quite meet the wants of those who have not all his experience and skill to guide their steps.

The difference between the series of places computed from the approximate elements and the true ones may be expressed by a series of the form

$$a + bt^2 + ct^2 + \&c.$$